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Letters

Computation of the Hecken Impedance Function

J. H. CLOETE

The Dolph–Chebyshev impedance function derived by Klopfenstein [1] has discontinuities at the taper ends which introduce unwanted effects in certain applications. The Hecken impedance function [2] is not optimum in the Dolph–Chebyshev sense, but achieves matching without impedance steps. For any bandwidth and maximum magnitude of reflection coefficient in the passband, the Hecken taper is only slightly longer than the optimum taper [2]. Hecken's near-optimum taper is therefore an attractive alternative to the optimum taper when impedance discontinuities are undesirable.

The equation for the near-optimum impedance function contains a transcendental function $G(B,\xi)$ which is tabulated in Hecken's paper. The function is given by

$$G(B,\xi) = \frac{B}{\sinh B}\int_0^{\xi} I_0\{B\sqrt{1-\xi'^2}\}\, d\xi'$$

where $I_0(z)$ is the modified Bessel function of the first kind and zero order.

Instead of using the tables, $G(B,\xi)$ may be computed recursively as

$$G(B,\xi) = \frac{B}{\sinh B}\sum_{k=0}^{\infty} a_k b_k$$

where

$$a_0 = 1 \quad a_k = \frac{B^2}{4k^2}\, a_{k-1}$$

$$b_0 = \xi \quad b_k = \frac{\xi(1-\xi^2)^k + 2kb_{k-1}}{2k+1}.$$

The derivation is based on the method described by Grossberg [3] and is not given here.

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Synthesis of Certain Transmission Lines Employed in Microwave Integrated Circuits

RAYMOND CRAMPAGNE AND GRATIA KHOO

With a quasi-TEM approximation, the characteristic parameters of numerous structures used as hyperfrequency microelectronics transmission lines can be calculated with the aid of conformal mapping. Simple theoretical formulas are rarely used since they bring into play the function $K(k)/K'(k)$ where $K(k)$ is the complete elliptic integral of the first type, $K'(k)$ its complementary function, and k its argument.

Some geometrical configurations which can be treated are shown in Fig. 1(a)–(c). This method is particularly interesting since expressions of k (argument of elliptic integral) as a function of geometric dimensions are often simple.

The infinite dielectric thickness hypothesis made in certain cases is, in general, justified by the spacing between conductors. Although this method is surprisingly simple accompanied by a large application domain, it has been put aside by many research workers. Instead, sophisticated numerical methods like those of finite differences and finite elements [1] have been preferred. These methods are applicable for the analysis of transmission lines but not for the synthesis. Moreover, they do not lead to

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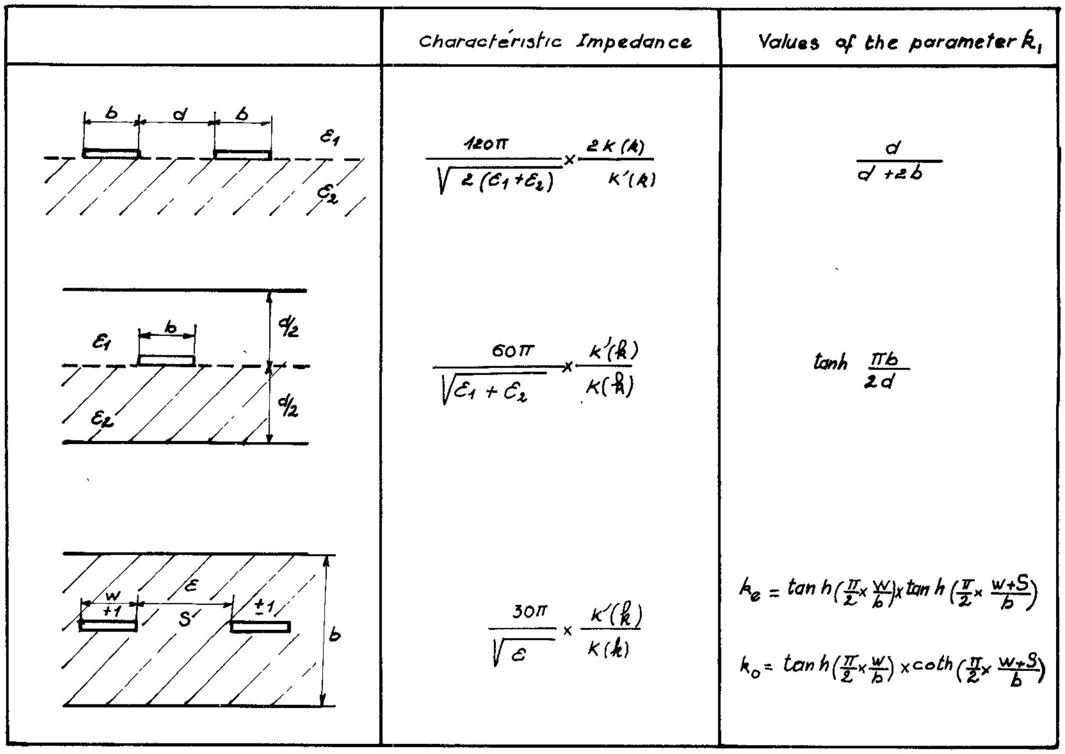

(a) (b) (c)

Fig. 1.

simple formulas which can be calculated with an electronic pocket calculator.

The aim is to obtain simple analytic formulas which may be employed in both the analysis and synthesis of lines where the precision of results obtained and the field of validity of these formulas are known.

With the aid of a classical algorithm [2], values of the function $f(k) = K(k)/K'(k)$ for all values of k between 0 and 1 have been obtained with a precision of 10^{-8}. A comparison of our results with those given in the tables [3] (for values of the argument k, corresponding to those of the tables) is a proof that the precision obtained is actually 10^{-8}.

A polynomial expression was then employed in smoothing out the curve obtained previously by minimizing the calculated error with the aid of the least squares method. Depending on the use of calculated values, a development of $f(k) = K/K'$ as a power of k, or a development of $k = f^{-1}(K/K')$ as a power of K/K', have been obtained. In order to obtain an error inferior to 10^{-4} for all values of the parameter, a polynomial approximation for two different independent intervals is necessary. A single interval with the same precision would require a polynomial of a degree such that its use in practice becomes inconvenient and contrary to our aim.

Only the development of f^{-1} will be given since a study [4] of the formulas given in the References indicates that the analysis of lines can be carried out with good precision by using the formula

$$\frac{K(k)}{K'(k)} = \frac{\pi}{\log(1/q)} \tag{1}$$

where

$$q = \varepsilon + 2\varepsilon^5 + 15\varepsilon^9 + 150\varepsilon^{13} + 1707\varepsilon^{17}$$

TABLE I
VALUES OF COEFFICIENTS FOR THE TWO POLYNOMIAL APPROXIMATIONS OF $k = f^{-1}(K/K')$

	0 K / K' 0,5		0,5 K / K' 1
a_0	0,00913	a_0	- 0,0656
a_1	- 0,1352	a_1	- 0,3211
a_2	+ 0,2689	a_2	+ 1,676
a_3	- 0,1367	a_3	+ 0,5984
a_4	+ 5,523	a_4	- 1,849
a_5	- 5,087	a_5	+ 0,6705

and

$$2\varepsilon = \frac{1 - \sqrt{k'}}{1 + \sqrt{k'}}, \qquad k'^2 = 1 - k^2.$$

Moreover, this formula can be easily manipulated on an electronic pocket calculator.

The polynomial approximation to be found is of the form

$$k = \sum_{i=0}^{N} a_i (K/K')^i. \tag{2}$$

The choice of the polynomial's degree depends upon the desired precision. In the two intervals chosen for K/K': $0 < K/K' < 0.5$ and $0.5 < K/K' < 1$, the minimum value of the polynomial's degree N, for an error precision of less than 10^{-4}, has been found to be 5. Values of the coefficients a_i (are shown in Table I. The following transformation formulas are helpful in restricting K/K' to the interval (0,1)

$$K(k') = K'(k) \qquad K(k) = K'(k') \qquad k^2 + k'^2 = 1. \quad (3)$$

Equation (2) is valuable in practice, since it allows the synthesis of all transmission lines to be realized. A few of them are shown in Fig. 1.

In fact, for a given characteristic impedance, the corresponding geometric dimensions of the line can be calculated rapidly by means of an electronic pocket calculator. Moreover, since the two developments are valid for all values of the characteristic impedance, one can know immediately if the realization of the latter is possible or not for a chosen geometrical configuration.

Hence, provided that the dielectric interfaces of geometrical configurations do not present any difficulty in conformal mapping, analytical expressions obtained can be easily manipulated, without having to resort to tables.

We hope that these synthesis formulas will find a place in the bibliography and allow engineers to make use of elliptic integrals with less hesitation.

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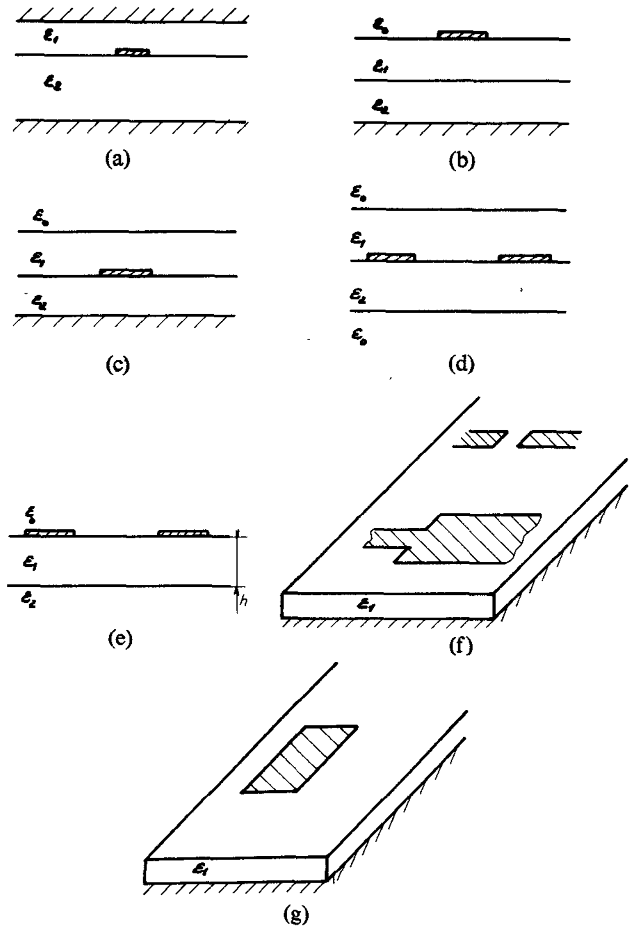

Fig. 1.

A Two- or Three-Dimensional Green's Function Which Can Be Applied to Hyperfrequency Microelectronic Transmission Lines

RAYMOND CRAMPAGNE AND JEAN-LOUIS GUIRAUD

Knowing Green's function and the charge density found on different conductors, the diverse capacities [1]–[4] can eventually be calculated by solving an integral equation. This has been dealt with only for simple dielectric-conductor configurations. In Coen's article [5], the integral representation of log (Z) is employed in calculating Green's function for microstrips (with or without an upper ground plane). Electrostatically speaking, the boundary conditions along conductors or dielectric interfaces are represented by means of infinite charge series.

We will treat two- or three-dimensional problems in exactly the same way; microstrip [Fig. 1(b) and (c)], triplate [Fig. 1(a)], and coplanar [Fig. 1(d) and (e)] types of transmission lines in a quasi-TEM approximation can be treated in the two-dimensional case [1], [2]. As indicated in Fig. 1, these lines can also be composed of several dielectrics. The three-dimensional case is employed in the calculation of capacitances or inductances of equivalent circuits representing discontinuities of certain lines [Fig. 1(f)] or the capacitances obtained by the localized element technique [Fig. 1(g)].

The aim is to find an integral representation of Green's functions in space with several dielectrics: log (Z) or $1/r$ depending upon whether the Green's functions in free space are a two- or three-dimensional problem. $Z = y + jx$ is a point in the Z plane which represents the cross section of a line charge; $r^2 = \rho^2 + u^2$ represents the distance between the point field and the point source of a point charge.

In the case of a homogeneous dielectric body of permittivity ε, the integral representation of Green's function for a line charge [5] situated at $x = 0$, $y = \alpha$ or a point charge [6] situated at $\rho = 0$, $u = \alpha$ can be written as

$$\phi(x,y) = \frac{1}{2\pi\varepsilon}\int_0^\infty \frac{\exp\,[-\lambda|y-\alpha|]\cos\lambda x - \exp\,[-\lambda]}{\lambda}\,d\lambda$$

or

$$\phi(\rho,u) = \frac{1}{4\pi\varepsilon}\int_0^\infty J_0(\lambda\rho)\exp\,[-\lambda|u-\alpha|]\,d\lambda. \quad (1)$$

All further developments will be based upon the following remark: A multiplication within the integral of expressions [1]

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